STABILITY CRITERION

MJS 11/4/03

$M = M_n M_{n-1} \cdots M_2 M_1$ describes a periodic section (could be 1 revolution!)

Stable oscillation $\longrightarrow$ $M^k \vec{X}_0$ must remain finite as $k \to \infty$

$\vec{X}_0 = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$ some initial condition

let $\vec{X}_0 = A\vec{V}_1 + B\vec{V}_2$, $\vec{V}_1, \vec{V}_2$ eigenvectors of M; A, B constants

then, $M^k \vec{X}_0 = A\lambda_1^k \vec{V}_1 + B\lambda_2^k \vec{V}_2$ λ_1, λ_2 eigenvalues

note: $\det M = 1$ [$\det M_1 = \det M_2 = \det M_n = 1$, and $\det M = \det M_1 \det M_2 \cdots \det M_n$] since energy of particle conserved

and $\det M = \lambda_1 \lambda_2$ $\therefore \boxed{\lambda_2 = \frac{1}{\lambda_1}}$ so, write $\lambda_1 = e^{i\mu}$, $\lambda_2 = e^{-i\mu}$

μ is complex, in general; μ real $\rightarrow$ STABILITY

Now, find the eigenvalues ...

let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ eigenvalue equation: $\det[M - I\lambda] = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$\boxed{(ad-bc) - (a+d)\lambda + \lambda^2 = 0}$$

$ad - bc = \det M = 1$; $a + d = \mathrm{Tr}M$

$$\Rightarrow \frac{1}{\lambda} + \lambda = \mathrm{Tr}M \quad \text{or} \quad \underbrace{e^{-i\mu} + e^{i\mu}}_{2\cos\mu,\ \text{if } \mu \text{ is real}} = \mathrm{Tr}M$$

$\therefore \boxed{|\mathrm{Tr}M| < 2}$ for stable motion