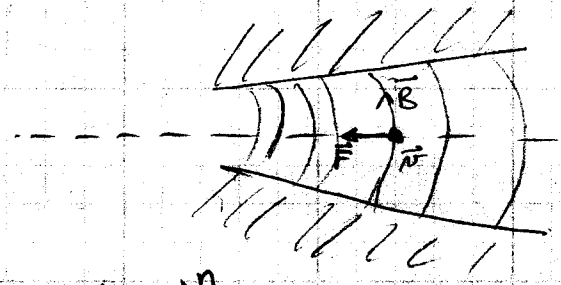
Transverse Motion ... 1

Need to force particles back toward the ideal trajectory

∴ would like a force ∝ displacement

early accelerators used "weak focusing" gradient magnets



$$B_y = B_0 \left(\frac{R_0}{r}\right)^n \qquad r = R_0 + x$$

$$= B_0 \left(1 + \frac{x}{R_0}\right)^{-n} \approx B_0 \left(1 - \frac{n}{R_0} x\right)$$

$$\therefore B_x \approx -n \frac{B_0}{R_0} y \qquad \left[\text{since } \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}\right]$$

system is stable (x + y) if $\underline{0 < n < 1}$.

Modern accelerators (HEP) use "strong focusing" (Courant, et al.) ...

Can separate Bending + Focusing functions using different magnets: dipoles, quadrupoles

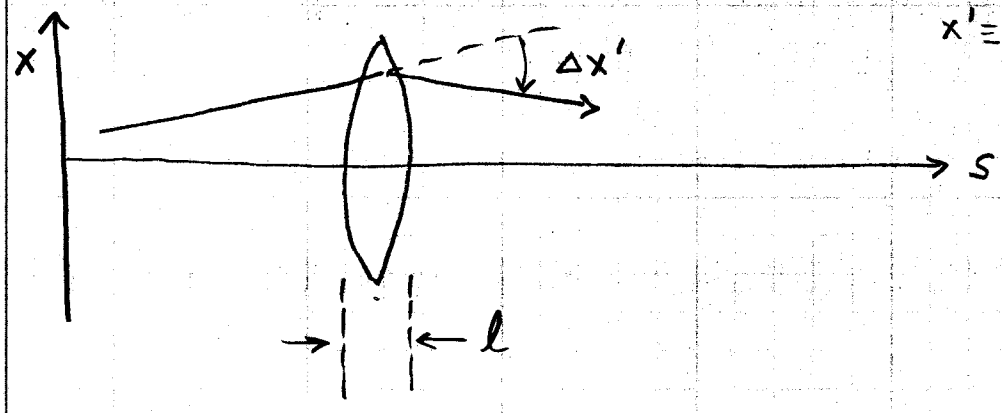
dipole: $B_y = B$ $\qquad \left[B = \frac{2\mu_0 NI}{d}\right]$

$B_x = 0$

quadrupole: $B_y = B'x$ $\qquad \left[B' = \frac{2\mu_0 NI}{R^2}\right]$

$B_x = B'y$

quadrupoles act like lenses ...



$x' \equiv \frac{dx}{ds}$ = slope

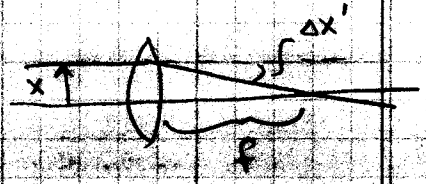
$$\Delta x' = \frac{\Delta p_x}{p} = \frac{\int F_x dt}{p} = -\frac{\int ev B_y dt}{p} \approx -\frac{e\int B_y ds}{p}$$

$$\approx -\frac{eB'l}{p} \cdot x$$

ratio $\frac{p}{e} \equiv$ "magnetic rigidity" $\equiv (Bp)$

$$(Bp) \approx \frac{10}{3} \cdot p_{[GeV/c]} \quad \text{Tesla-m}$$

So, $\Delta x' = -\underbrace{\frac{B'l}{Bp}}_{\frac{1}{f}} \cdot x$



"thin lens" approximation : valid if $l \ll f$

ex: Tevatron @ 980 GeV :

$(Bp) = \frac{10}{3} \cdot 980$ T·m = 3270 T·m

$B' = 78$ T/m $l = 1.68$m

and $f = \frac{(Bp)}{eB'l} = \frac{3270 \text{ T·m}}{(78 \text{ T/m})(1.68\text{m})} = 25\text{m} \gg l$

note: $B_y = B'x$, $B_x = B'y$ } quadrupole field

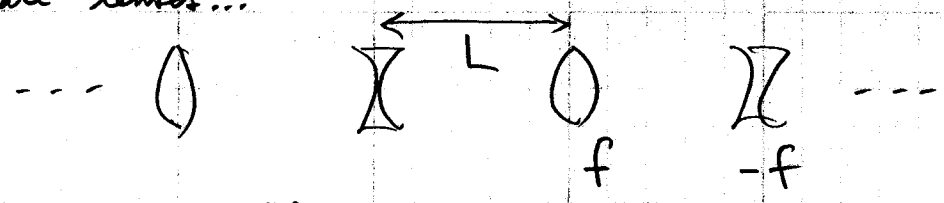
$\vec{F} = e\vec{v} \times \vec{B} \Rightarrow$ $F_x = -evB_y = -evB'x$

$F_y = +evB_x = +evB'y$

$\therefore B' > 0 \Rightarrow$ focuses in x-direction

defocuses in y-direction

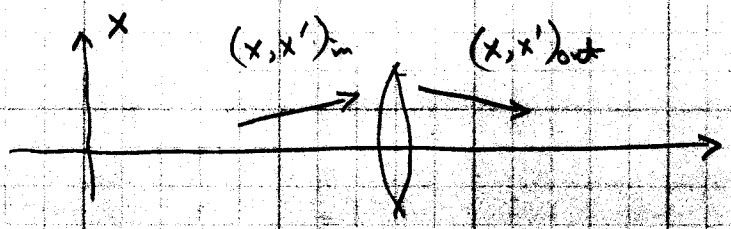
alternate lenses...



system is stable (x & y)

if $\boxed{f > L/2}$

2×2 matrices...



quad (thin lens): $\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$

drift (or bend region): $\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$

∴ build up accelerator out of pieces, each piece described by 2×2 matrix (actually, 2 (x,y) matrices)

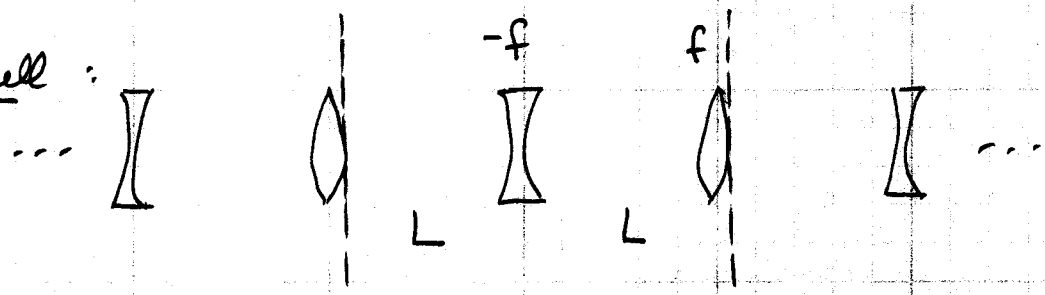
then, total ring described by

$$M = M_n M_{n-1} \cdots M_2 M_1$$

motion is stable if $\boxed{|\text{Trace } M| < 2}$

stability criterion

FODO cell:



$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & L \\ -\frac{1}{f} & 1-\frac{L}{f} \end{pmatrix}\begin{pmatrix} 1 & L \\ \frac{1}{f} & 1+\frac{L}{f} \end{pmatrix} = \begin{pmatrix} 1+\frac{L}{f} & 2L+\frac{L^2}{f} \\ -\frac{L}{f^2} & 1-\frac{L}{f}-\frac{L^2}{f^2} \end{pmatrix}$$

$$\text{trace } M = 2 - \frac{L^2}{f^2}$$

∴ criterion: $-2 < 2 - \frac{L^2}{f^2} < 2$

$$-4 < -\frac{L^2}{f^2} < 0$$

or, $\frac{L^2}{f^2} < 4 \Rightarrow \boxed{f > \frac{L}{2}}$

can always solve specific problem with a "piece-wise" solution; next, we look at a more general, analytical approach...

System is analogous to SHM, but the "spring constant" depends on time (or, here, on longitudinal position)...

$$\ddot{X} + kx = 0 \quad \Rightarrow \quad x = A\sin(\omega t + \delta)$$

we have

$$x'' + K(s)x = 0 \quad \Rightarrow \quad x(s) = A(s)\sin\left[\int \omega(s)dt + \delta\right]$$

our "spring constant":

$$\Delta x' = -\frac{B'\Delta s}{B\rho}\cdot x \quad \Rightarrow \quad x'' = -\frac{B'}{B\rho}x \qquad \text{(quad. field)}$$

in fact, we also have a "rotating" reference system (bending magnets), so we have

$$K_x(s) = \frac{\frac{\partial B_y(s)}{\partial x}}{(B\rho)} + \frac{1}{\rho(s)^2}$$

$$K_y(s) = \frac{\frac{\partial B_y(s)}{\partial x}}{B\rho}$$

ρ = local bending radius (typically very large!)

but, in each transverse degree of freedom we have

$$\boxed{x'' + K(s)x = 0}$$

Hill's Equation

Trial Solution: $x(s) = A\sqrt{\beta(s)}\,\sin[\psi(s) + \delta]$

from here, take $x'(s) = \frac{1}{2}A\beta^{-1/2}\beta'\sin[\,] + A\beta^{1/2}\cos[\,]\psi'$

and take $x''(s) = \ldots$

+ plug into Hill's Equation

we find that we must have

$$\psi' = \frac{1}{\beta}, \quad \text{and} \quad K\beta = -\frac{1}{2}\beta'' + \frac{1+\frac{1}{4}(\beta')^2}{\beta}$$

$\psi' = \frac{1}{\beta}$ ⇓

diff. eq. that β must obey

β is related to local amplitude of motion

AND

β is a local wavelength of the motion $(\lambda/2\pi)$

So, β is called the Amplitude Function. The oscillation phase advances according to

$$\Delta\psi = \int \frac{1}{\beta(s)}\,ds$$

Total # of oscillations around the accelerator:

$$\nu = \frac{1}{2\pi}\oint \frac{1}{\beta}\,ds \equiv \text{the "tune"}$$

note: $\Delta\varphi = \int \frac{ds}{\beta}$ $\Longrightarrow$ β has units of length

also, since oscillation phase builds up over large distance, β is a relatively large number

particle's oscillation amplitude $\propto A\sqrt{\beta}$, so A must have units of length$^{1/2}$ (more later)

$\sqrt{\beta}$ describes the form of the envelope of a particle's motion; the arbitrary constant A depends upon the particle's initial conditions

(BASIC PROGRAM

So, how to compute $\beta(s)$...

a) solve $K\beta + \frac{1}{2}\beta'' - \frac{1+\frac{1}{4}(\beta')^2}{\beta} = 0$ (hard)

actually, $\beta\beta'' + 2K\beta^2 - \frac{1}{2}\beta'^2 - 2 = 0$

~~$\beta'\beta'' + \beta\beta''' + 2K'\beta^2 + 4K\beta\beta' - \beta'\beta''$~~ $= 0$

$$\boxed{\beta''' + 4K\beta' + 2K'\beta = 0}$$ (easier)

or,

b) relate $\beta(s)$ to our piece-wise matrix methods (typical)

i.e., write

$$M = \begin{pmatrix} a(\beta,\beta') & b(\beta,\beta') \\ c(\beta,\beta') & d(\beta,\beta') \end{pmatrix}$$

and compare numerically to the piecewise solution

take $X = A\sqrt{\beta}\sin(\psi + \delta)$

and re-write $X = A\sqrt{\beta}\sin\psi + B\sqrt{\beta}\cos\psi$

find $x' = \ldots$

assume $X = X_0$, $X' = X_0'$ at point S_0,
and write in matrix form:

$$\begin{pmatrix} X(s) \\ x'(s) \end{pmatrix} = M_{\text{from } s_0 \text{ to } s} \begin{pmatrix} X_0 \\ X_0' \end{pmatrix}$$ to get...

$$M_{\text{from } s_0 \text{ to } s} = \begin{pmatrix} \sqrt{\beta/\beta_0}(\cos\psi + \alpha_0\sin\psi) & \sqrt{\beta\beta_0}\sin\psi \\ -\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\psi + \frac{\alpha_0-\alpha}{\sqrt{\beta\beta_0}}\cos\psi & \sqrt{\frac{\beta_0}{\beta}}(\cos\psi - \alpha\sin\psi) \end{pmatrix}$$

where $\psi = \psi(s) - \psi(s_0)$, and $\boxed{\alpha \equiv -\frac{1}{2}\beta'}$

Courant-Snyder Parameters

β = amplitude function

$\alpha \equiv -\frac{1}{2}\beta'$

$\gamma \equiv \frac{1+\alpha^2}{\beta}$

note: $\boxed{K\beta = \alpha' + \gamma}$ diff eq, short form!

if go once around the accelerator, then $s = s_0 + C$
and we demand that β be periodic solution...

$$\rightarrow \quad M_{s_0 \to s_0 + C} = \begin{pmatrix} \cos 2\pi\nu + \alpha \sin 2\pi\nu & \beta \sin 2\pi\nu \\ -\gamma \sin 2\pi\nu & \cos 2\pi\nu - \alpha \sin 2\pi\nu \end{pmatrix}$$

note: $\boxed{\text{trace } M_0 = 2\cos 2\pi\nu}$

$\therefore$ can extract the fractional part of the tune

if $M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ numerically, then

$$\beta(s_0) = \frac{b}{\sin 2\pi\nu}$$

(note: chose $\sin 2\pi\nu$ so that $\beta > 0$)

... etc.

ex: periodic structure of FODO cells ($@$ s_0, just after F quad):

$$M_0 = \begin{pmatrix} 1 + \frac{L}{f} & 2L + L^2/f \\ -L/f^2 & 1 - L/f - L^2/f^2 \end{pmatrix}$$

$$\text{trace } M_0 = 2 - L^2/f^2 = 2\cos\mu$$

$$\Rightarrow \boxed{\sin \mu/2 = L/2f}$$

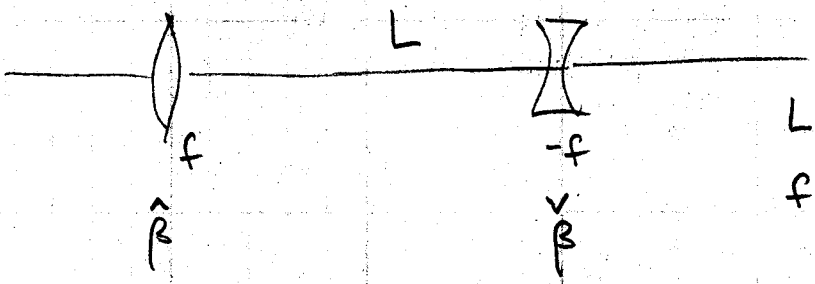
here, μ = phase advance thru 1 cell

if only FODO cells, $2\pi\nu = N_{cells} \cdot \mu$

$$\beta = \frac{2L + L^2/f}{\sin\mu} = \frac{2L\,(1 + L/2f)}{2\sin\frac{\mu}{2}\cos\frac{\mu}{2}} = 2f\,\frac{1 + \sin\mu/2}{\sqrt{1 - \sin^2\mu/2}}$$

$$\text{or} \quad \boxed{\beta = 2f\sqrt{\frac{1+\sin\mu/2}{1-\sin\mu/2}}}$$

example: TEVATRON



$L = 30\,m$

$f = 25\,m$

$$\mu = 2\sin^{-1}\left(\frac{L}{2f}\right) = 2\cdot\sin^{-1}\left(\frac{30}{50}\right) \approx \frac{30}{25}\ \text{rad} \approx 68^\circ$$

$$\hat{\beta} = 2\cdot 25m\cdot\sqrt{\frac{1+0.6}{1-0.6}} = 100\ m$$

$$\check{\beta} = 2\cdot 25m\sqrt{\frac{1-0.6}{1+0.6}} = 25\ m$$

note: $\nu \approx \frac{1}{2\pi} N_{cells}\cdot\mu \approx \frac{1}{2\pi}\,100\cdot\frac{6}{5} \approx 20$

A shorthand notation:

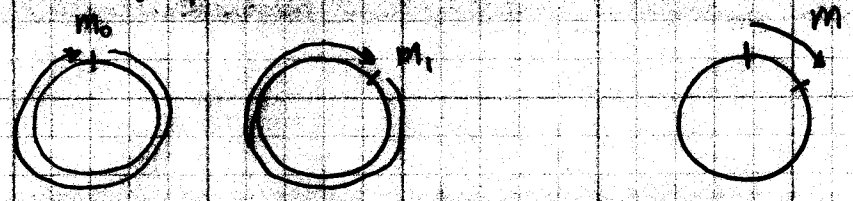
$$M = I\cos\mu + J\sin\mu = e^{J\mu}$$

$$I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$J^2 = -I$$

where $J = \begin{pmatrix}\alpha&\beta\\-\gamma&-\alpha\end{pmatrix}$

Transportation of β...



$$m_1 = m\,m_0\,m^{-1}$$

so, $I\cos\mu + J_1\sin\mu = m\,(I\cos\mu + J_0\sin\mu)\,m^{-1}$

$$= I\cos\mu + m\,J_0\,m^{-1}\sin\mu$$

so, $\boxed{J_1 = m\,J_0\,m^{-1}}$

HW problems

a) Find the angular deflection $\Delta\theta$ generated by a fast rise-time kicker magnet of length 2m and field strength 0.025 T acting on a 150 GeV/c proton.

b) Assume the Tevatron is made of 100 identical FODO cells.

i) find β, α, γ, and μ for a cell which begins just after a focusing quadrupole.

ii) write out the matrix M_{RING} representing passage through all 100 cells. [here, $M_{RING} = (M_{cell})^{100}$]

iii) if a proton is given a sudden kick by the amount $\Delta\theta$ found in a), i.e., if $x_0' = \Delta\theta$ and $x_0 = 0$, then find its resulting maximum displacement in our 100-FODO-cell ring.

c) For our "100 FODO cell TEVATRON" what $\frac{\Delta B'}{B'}$ is required in all the quadrupoles to change the tune by one unit (i.e., from ~20 to ~21, say)